Area spectrum of rotating black holes via the new interpretation of quasinormal modes

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Motivated by the recent work on a new physical interpretation of quasinormal modes by Maggiore, we utilize this new proposal to the interesting case of Kerr black hole. In particular, by modifying Hod's idea, the resulting black hole horizon area is quantized and the resulting area quantum is in full agreement with Bekenstein's result. Furthermore, in an attempt to show that the area spectrum is equally spaced, we follow Kunstatter's method. We propose a new interpretation as a result of Maggiore's idea, for the frequency that appears in the adiabatic invariant of a black hole. The derived area spectrum is similar to that of the quantum-corrected Kerr black hole but it is not equally spaced.

Since the onset of General Relativity black holes have been a matter of major concern for the scientific community. This interest is twofold. On one hand, black holes are astrophysical objects whose fingerprints will be observed on recent or future detectors for gravitational waves e.g. LIGO [1] and VIRGO [2]. On the other hand, black holes have always been a test bed for any proposed scheme for a quantum theory of gravity. It is evident that it would be of great importance for quantum gravity (and not only) if the superficially distinct (astrophysical vs theoretical) aspects could be reconciled. Hod was one of the first to make such a phenomenological work [3]. He combined the perturbations of astrophysical black holes with the principles of Quantum Mechanics and Statistical Physics in order to derive the quantum of the black hole area spectrum. Following this line of thought, Kerr black holes are the most interesting black hole solutions since from the astrophysical point of view are the most important ones while from the purely theoretical point of view are more complicated than the simple Schwarzschild black hole.The metric of a four-dimensional Kerr black hole given in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 &= -(1-\frac{2Mr}{\Sigma})dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\varphi + \frac{\Sigma}{\Delta}dr^2 \\ &+ \Sigma d\theta^2 + (r^2+a^2+2Ma^2r\sin^2\theta)\sin^2\theta d\varphi^2 \end{aligned} \quad (1)$$

where, as always, M is the mass of the black hole, J is the angular momentum of the black hole, a is the specific angular momentum defined as J/M, $\Sigma = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$. The roots of Δ are given by

$$r_\pm = M \pm \sqrt{M^2 - a^2} \quad (2)$$

where r_+ is the radius of the event (outer) black hole horizon and r_- is the radius of the inner black hole horizon. The Kerr black hole is rotating with angular velocity (evaluated on the event black hole horizon)

$$\Omega = \frac{a}{r_+^2 + a^2} = \frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)} . \quad (3)$$

Furthermore, the horizon area and the Hawking temperature of Kerr black hole (in gravitational units) are given, respectively, by

$$A = 4\pi(r_+^2 + a^2) = 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right) \quad (4)$$

and

$$T_H = \frac{r_+ - r_-}{A} = \frac{\sqrt{M^4 - J^2}}{4\pi M\left(M^2 + \sqrt{M^4 - J^2}\right)} . \quad (5)$$

As mentioned before, Hod managed to derive the quantum of the area spectrum using the Bohr's Correspondence principle and the complex spectrum of the quasinormal modes that correspond to the perturbation equation of Schwarzschild black hole. The resulting quantum was of the form [3]

$$\Delta A = 4l_p^2 \ln 3 \quad (6)$$

where l_p is the Planck length. Hod's idea[1] rejuvenated the interest of the research community for the quantization of the black hole area spectrum and subsequently for a derivation of black hole entropy from Statistical Physics. Actually, the aroused interest was strengthened by the possible links with loop quantum gravity as proposed by Dreyer[2] [7].

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[1] Later, it was shown by Natario and Schiappa [4] that Hod's calculation is not universal since it depends on the asymptotics of the black hole spacetime under study.

[2] However, it should be emphasized that the method used by

Some thirty five years ago, by proving that the black hole horizon area is an adiabatic invariant, Bekenstein showed that the quantum of black hole area is of the form [8]

$$\Delta A = 8\pi l_p^2 \ . \tag{7}$$

Adiabatic invariants of a system are quantities which vary very slowly compared to variations of the external perturbations of the system. Moreover, given a system with energy E and vibrational frequency $\omega(E)$, one can show that the quantity E/ω and therefore

$$I = \int \frac{dE}{\omega(E)} \ , \tag{8}$$

is an adiabatic invariant. For the case of black holes, as already said above, Bekenstein was the first to state that the adiabatic invariants are the black hole horizon areas [9, 10].

Exploiting the idea of adiabatic invariants and the statement by Bekenstein [8], Kunstatter [11] derived for the $d(\geq 4)$-dimensional Schwarzschild black hole an equally spaced entropy spectrum. Key points to Kunstatter's approach were :

(1) the first law of black hole thermodynamics which for the case of a Schwarzschild black hole is of the form

$$dM = \frac{1}{4} T_H dA \ , \tag{9}$$

(2) Hod's proposal that in the asymptotic limit, i.e. the large n limit, the real part of quasinormal frequencies of the Schwarzschild black hole uniquely fixes the quantum of the black hole area spectrum, and

(3) the fact that the Bohr-Sommerfeld quantization has an equally spaced spectrum in the large n limit, i.e.

$$I \approx n\hbar \ . \tag{10}$$

Kunstatter viewed the Schwarzschild black hole as a system whose adiabatic invariant takes the form

$$I = \int \frac{dM}{\omega_R} \tag{11}$$

where dE was set equal to dM and the frequency in the denominator of the integral in equation (8) was set equal to the real part of the quasinormal frequencies of the Schwarzschild black hole which was $\omega_R \sim T_H$. Finally, the area spectrum and thus the entropy of the Schwarzschild black hole were discrete and equally spaced. At that point Kunstatter raised the interesting question if the aforesaid derivation holds for rotating black holes. In this direction, Hod studied analytically the quasinormal modes of Kerr black hole [12] and he concluded that the asymptotic quasinormal frequencies of Kerr black hole are given by the simple expression

$$\omega = m\Omega - i2\pi T_H n \tag{12}$$

which were in agreement for the case of $l = m = 2$ with the numerical results derived by Berti and Kokkotas [13].

Endeavoring to answer Kunstatter' question we extended his approach [14] to the case of Kerr black hole using the real part of the quasinormal frequency given in equation (12). The first law of black hole thermodynamics is now written as

$$dM = \frac{1}{4} T_H dA + \Omega dJ \tag{13}$$

where the angular velocity is given by equation (3) and obviously the corresponding expression for adiabatic invariant is now given by the expression

$$I = \int \frac{dM - \Omega dJ}{\omega_R} \ . \tag{14}$$

Equating Bohr-Sommerfeld quantization condition (10) with the adiabatically invariant integral (14) one obtains an area spectrum for the Kerr black hole which although discrete, is not equidistant. However, it was proven by Bekenstein [8, 9] and others [15, 16] that the area spectrum of Kerr black hole is discrete and uniformly spaced. Therefore, it was concluded that the function that was used in the above-mentioned computation as real part of the asymptotic quasinormal frequencies of Kerr black hole, i.e. expression (12), was not the correct one. Recent analytical works [17, 18] confirmed older numerical calculations [19] in which the quasinormal frequencies of a Kerr black hole are of the form

$$\omega(n) = \tilde{\omega}_0 - i\left[4\pi T_0 \left(n + \frac{1}{2}\right)\right] \tag{15}$$

where $\tilde{\omega}_0$ is a function of the black hole parameters and T_0 is the effective temperature. For $M^2 \gg J$, or equivalently $a/M \approx 0$, the effective temperature is

$$T_0(a) \approx -\frac{T_H(a=0)}{2} \tag{16}$$

and $T_H(a = 0)$ is the Hawking temperature of the Schwarzschild black hole (henceforth T_H^{Sch}). The subscripts of the frequency $\tilde{\omega}$ and temperature T in equation (15) denote that these quantities have been computed by integrating a contour that crosses the real axis outside the event horizon [18].

Very recently a new physical interpretation for the quasinormal modes of black holes was given by Maggiore [20]. According to Maggiore's proposal if one wants

Dreyer for state counting was incorrect and consequently, Dreyer computed an incorrect value for the Barbero-Immirzi parameter. The correct method of counting states was proposed by Domagala and Lewandowski [5]. Furthermore, implementing the correct method Meissner calculated the correct value for Barbero-Immirzi parameter [6] which was between the upper and lower bounds set in [5].

to avoid several problems in the interpretation of quasinormal frequencies when compared with macroscopical systems, one has to treat a perturbed black hole as a damped harmonic oscillator. Then one has to identify as proper frequency of the equivalent harmonic oscillator the following quasinormal normal frequency

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2} \tag{17}$$

which decidedly for the case of long-lived quasinormal modes, i.e. $\omega_I \to 0$, the frequency of the harmonic oscillator becomes $\omega_0 = \omega_R$. However, the most interesting case is that of highly excited quasinormal modes for which $\omega_I \gg \omega_R$ and thus the frequency of the harmonic oscillator becomes $\omega_0 = \omega_I$. Furthermore, Maggiore proposed that if one wants to solve or at least alleviate problems that were raised by the Hod's proposal one has to employ the ω_0 rather than ω_R since in order to derive the quantum spectrum of a black hole using its quasinormal modes, the black hole has to be treated as a collection of damped harmonic oscillators. In this framework, we consider the transition $n \longrightarrow n-1$ for a Kerr black hole. Since we are interested in highly excited black holes, i.e. n is large, the proper frequency is now $\omega_0 = \omega_I$ and thus the absorbed energy using equations (15) and (16) is

$$\begin{aligned} \Delta M &= \hbar\left[(\omega_0)_n - (\omega_0)_{n-1}\right] \\ &= \hbar\left[(\omega_I)_n - (\omega_I)_{n-1}\right] \qquad (18) \\ &= -4\pi\hbar T_0 = 2\pi\hbar T_H^{Sch} \ . \qquad (19) \end{aligned}$$

This change in the black hole mass will create a change in the black hole area of the form

$$\Delta A = 32\pi M \Delta M \tag{20}$$

and substituting the change of black hole mass as given by equation (19), the change in the black hole area becomes

$$\Delta A = 8\pi\hbar = 8\pi l_p^2 \ . \tag{21}$$

A couple of comments are in order here. First, our result for the Kerr black hole is in full agreement with that for the Schwarzschild black hole given by Maggiore. Second, we have managed to derive a universal area quantum, i.e. independent of the parameters that characterize the Kerr black hole. Therefore, the concept of universality for the area quantum has from now on a twofold meaning. On one hand, it means that the quantum of the area spectrum is independent of the black hole parameters and on the other hand, it means that it is the same for the Schwarzschild and Kerr black hole. It should be stressed that the two meanings are interwoven since the first statement in the limit $a \to 0$ (which reduces the Kerr black hole to the Schwarzschild black hole) leads us directly to the second one, and the other way around. It is noteworthy that the change in the area of Kerr black hole (20) is that of the Schwarzschild black hole. The reason for that is the fact that we are interested in highly damped quasinormal modes where as stated before $\omega_I \gg \omega_R$. This condition implies that $M^2 \gg J$ and therefore the angular part in the formula for the horizon area change can be neglected. The same condition holds for the effective temperature (16) of the quasinormal frequency spectrum (15). It seems that the relaxation time $\tau = \omega_I^{-1}$ is adequate for the damping to "wash out" the change in the angular momentum (ΔJ) but not the change in the mass (ΔM).

Let us now try to derive the quantized area spectrum of the Kerr black hole employing Kunstatter's method. Implementing the first law of black hole thermodynamics (13), the adiabatically invariant integral (8) is now given as

$$I = \int \frac{dM - \Omega dJ}{\omega} \ . \tag{22}$$

At this point one has to clarify what the frequency ω in the denominator should be. For the case of a harmonic oscillator, we claimed that this frequency is the vibrational frequency that corresponds to the system's energy E for which under a slow variation of a parameter which is related to the energy, a small variation dE in the energy was created and the quantity E/ω is an adiabatic invariant. Following Maggiore's proposal the perturbed black hole is treated as a set of harmonic oscillators. In the context of this correspondence, one has to define the cause for the small variations in the mass (ΔM) and the angular momentum (ΔJ) of the Kerr black hole. According to our previous syllogism, it is evident that for the case of black holes it is the transitions of type $n \longrightarrow n-1$, where $n \gg 1$, which make the black hole mass and angular momentum vary slowly and thus alter the entropy of the black hole through the first law of black hole thermodynamics. Therefore, the small variations in the mass and angular momentum of the black hole stem from the transitions and for this reason the frequency ω should be the one that corresponds to the absorbed energy given by equations (18) and (19), i.e. the transition frequency

$$\begin{aligned} \omega &= \left[(\omega_I)_n - (\omega_I)_{n-1}\right] \qquad (23) \\ &= 2\pi T_H^{Sch} \ . \qquad (24) \end{aligned}$$

Therefore, the adiabatic invariant for the Kerr black hole is now written as

$$\begin{aligned} I &= \int \frac{dM - \Omega dJ}{\left[2\pi T_H^{Sch}\right]} \qquad (25) \\ &= \Big[2M^2 + 2\sqrt{M^4 - J^2} \\ &- \ 2M^2 \log\left(M^2 + \sqrt{M^4 - J^2}\right)\Big] \ . \qquad (26) \end{aligned}$$

Using the expression for the Kerr black hole horizon area (4), the adiabatic invariant is rewritten as

$$I = \left[\frac{A}{4\pi} - 2M^2 \log\left(\frac{A}{8\pi}\right)\right] \tag{27}$$

and implementing the Bohr-Sommerfeld quantization condition (10), the quantized area spectrum is

$$\mathcal{A}_n = 4\pi l_{Pl}^2 n \tag{28}$$

where $\mathcal{A}_n$ is similar but not the same with the quantum-corrected Kerr black hole horizon area due to logarithmic corrections (see for instance [21]). The difference stems to the fact that the logarithmic "prefactor" α is not of order unity but depends on the black hole mass, i.e. $\alpha \approx M/l_{Pl}$. The negative sign that accompanies the logarithmic "prefactor" denotes that the logarithmic correction is of microscopic nature. More importantly, it should be stressed that since we are working with the highly damped quasinormal modes, i.e. in the large n limit, the "prefactor" transforms the logarithmic correction into the dominant term. This leads to a non-equidistant area spectrum [3] .

At this point a couple of comments are in order. First, after the present work Medved showed that if one takes the limit $M^2 \gg J$ (which we introduced for the derivation of the quantum of the area spectrum, i.e. equation (21)) into account for the computation of the integral in equation (25), then one ends up with an evenly spaced area spectrum [22]. Second, it is noteworthy that similar arguments for use of the imaginary part of the quasinormal frequencies were presented by Kiselev [23]. In addition, Kiselev showed that the area spectrum for the case of extremal Kerr black hole was identical with the one given here by equation (28), while for the non-extremal Kerr black hole the results were significantly different compared to the ones derived in the present analysis.

We have succeeded in deriving the quantum of the area spectrum of Kerr black hole adopting the new physical interpretation for the black hole quasinormal modes. This provides a strong evidence in support of the correctness of Maggiore's proposal. The area quantum is characterized by universality which has a twofold meaning: (α) the area quantum of Kerr black hole horizon is independent of its parameters, i.e. the mass M and the angular momentum J, and (β) the area quantum of Kerr black hole horizon is identical with the area quantum of the Schwarzschild black hole as derived by Maggiore. In addition, it is worth noting that the derived area quantum of Kerr black hole is the same with the one obtained by Bekenstein who employed the concept of adiabatic invariants. Finally, we proposed a new interpretation for the frequency of the adiabatic invariant of a black hole that appears in the context of Kunstatter's method. This new interpretation was introduced in order the concept of adiabatic invariant to be incorporated in Maggiore's proposal. This identification combined with the new interpretation of Maggiore led us to obtain in this context the quantized area spectrum of the Kerr black hole. However it failed to give an equidistant area spectrum since the limit $M^2 \gg J$ was not employed.

[3] We thank Allan Medved for indicating this subtle point.

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